Jour. Ind. Soc. Ag. Statistics
Vol. XXXVII, No. 3 (1985), pp. 271-275

ON THE MULTIVARIATE ANALOGUE OF SEQUENTIAL SIMULTANEOUS ESTIMATION PROBLEM

AJIT CHATURVEDI
Lucknow University, Lucknow

(Received : December, 1984)

SUMMARY

Sequential procedures are proposed for simultaneous estimation of the mean vector and scalar multiplier of covariance matrix of a p-variate normal population. Asymptotic behaviours of the procedures are studied.

Keywords : p-variate normal population; Euclidean space; Loss function.

Introduction

Mukhopadhyay [2] developed sequential procedures for simultaneous estimation of the mean and variance of a univariate normal population. He constructed a semi-circular region of given maximum diameter, which covers these parameters with prescribed confidence coefficient. Sequential point estimation procedure (the loss being quadratic) was also discussed.

In the present article, a multivariate extension of Mukhopadhyay's procedure is given. The population to be sampled is a p-variate normal population $N_p(\underset{\sim}{\mu}, \sigma^2 I_p)$, where $\underset{\sim}{\mu}$ is unknown mean vector, σ^2 is unknown scalar, and I_p stands for a $p \times p$ identity matrix. Thus the problem is to estimate $\underset{\sim}{\theta} = (\underset{\sim}{\mu}, \sigma^2)'$.

Let $\{\underset{\sim}{X}_i\}$, $i = 1, 2, \ldots$ be a sequence of independent random observations from $N_p(\underset{\sim}{\mu}, \sigma^2 I_p)$. Having recorded a sample $(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$ of size n, define, for $n \geqslant 2$,

$$\bar{\mathbf{X}}_n = n^{-1} \sum_{i=1}^{n} \mathbf{X}_i$$

and

$$\hat{\sigma}_n^2 = (p(n-1))^{-1} \sum_{i=1}^{n} (\mathbf{X}_i - \bar{\mathbf{X}}_n)'(\mathbf{X}_i - \bar{\mathbf{X}}_n)$$

as the estimators for $\underset{\sim}{\mu}$ and σ^2, respectively. It is easy to verify that these estimators are unbiased and consistent for the corresponding parameters. Moreover, the variance covariance matrix of $\bar{\mathbf{X}}_n$ is $(\sigma^2/n)I_p$ and $p(n-1)\hat{\sigma}_n^2/\sigma^2$ is distributed as χ^2 with $p(n-1)$ degrees of freedom.

Given d, $\alpha (d > 0, 0 < \alpha < 1)$, suppose one wishes to construct a semi-circular region R_n in p-dimensional Euclidean space such that $P(\underset{\sim}{\theta} \in R_n) \geqslant \alpha$ and the diameter of R_n is less than or equal to $2d$. It is proposed

$$R_n = \{\mathbf{Z} = (\mathbf{a}, b)' : b > 0 \quad \text{and} \quad (\mathbf{Z}_n - \mathbf{Z})'(\mathbf{Z}_n - \mathbf{Z}) \leqslant d^2\},$$

where $\mathbf{Z}_n = (\bar{\mathbf{X}}_n, \hat{\sigma}_n^2)'$.

Now define a $(p+1) \times (p+1)$ positive definite matrix

$$Q = \begin{pmatrix} \sigma^2 I_p & 0 \\ 0 & 2\sigma^4/p \end{pmatrix}$$

and $\lambda = \max\{\sigma^2, 2\sigma^4/p\}$. It can be verified that the ellipsoid

$$R_n^* = \{\mathbf{Z} = (\mathbf{a}, b)' : b > 0 \quad \text{and} \quad \lambda(\mathbf{Z}_n - \mathbf{Z})'Q^{-1}(\mathbf{Z}_n - \mathbf{Z}) \leqslant d^2\}$$

is contained in R_n. Further,

$$P(\underset{\sim}{\theta} \in R_n^*) = P\{\bar{\mathbf{X}}_n - \underset{\sim}{\mu})'(\sigma^2 I_p)^{-1}(\bar{\mathbf{X}}_n - \underset{\sim}{\mu}) + (2\sigma^4/p)^{-1}(\hat{\sigma}_n^2 - \sigma^2)^2 \leqslant d^2/y\}$$

$$= P\left\{\frac{1}{n}\chi_p^2 + \frac{1}{n-1}\chi_1^2 \leqslant d^2/\lambda\right\}$$

$$\geqslant P\left\{\chi^2_{(p+1)} \leqslant \frac{d^2}{\lambda}(n-1)\right\} \qquad (1.1)$$

Let 'a' be any constant such that

$$P(\chi^2_{(p+1)} \leqslant a^2) = \alpha \qquad (1.2)$$

It is clear from (1.1) and (1.2) that for σ known, in order to achieve $P(\underset{\sim}{\theta} \in R_n^*) \geqslant \alpha$, the required sample size n is the smallest positive integer greater than or equal to n^*, where $n^* = 1 + (a/d)^2\lambda$.

However, in absence of any knowledge about σ, no fixed-sample size procedure serves the purpose. In such a situation adopt a sequential procedure which is discussed in the next section.

2. The Sequential Procedure

Let

$$\hat{Q}_n = \begin{pmatrix} \hat{\sigma}_n^2 I_p & 0 \\ 0 & 2\hat{\sigma}_n^4/p \end{pmatrix}$$

and $\hat{\lambda}_n = \max\{\hat{\sigma}_n^2, 2\hat{\sigma}_n^4/p\}$. The stopping rule is defined as follows.

The stopping time $N \equiv N(d)$ is the smallest positive integer $n \geqslant m$ (>2) such that

$$n \geqslant (a_n/d)^2\hat{\lambda}_n + 1, \tag{2.1}$$

where $\{a_n\}$ $n = 1, 2, \ldots$ is a sequence of positive constants, converging to 'a'. When stop, construct R_N for $\underset{\sim}{\theta}$.

Now establish the following theorem.

THEOREM 1. *N is well-defined, non-decreasing as a function of d, and*

$$\lim_{d\to 0} N = \infty \text{ a.s.} \tag{2.2}$$

$$\lim_{d\to 0} \frac{N}{n^*} = 1 \text{ a.s.} \tag{2.3}$$

$$\lim_{d\to 0} E\left(\frac{N}{n^*}\right) = 1 \tag{2.4}$$

$$\lim_{d\to 0} P(\underset{\sim}{\theta} \in R_N) \geqslant \alpha \tag{2.5}$$

Proof. Result (2.2) follows from the definition of N at (2.1).

Note the basic inequality

$$\left(\frac{a_N}{d}\right)^2\hat{\lambda}_N + 1 < N \leqslant \left(\frac{a_N - 1}{d}\right)^2\hat{\lambda}_{N-1} + 1 + m$$

or

$$\left(\frac{a_N}{a}\right)^2\frac{\lambda_N}{\lambda} + \frac{1}{n^*} < \frac{N}{n^*} \leqslant \left(\frac{a_{N-1}}{a}\right)^2\frac{\hat{\lambda}_{N-1}}{\lambda} + \frac{(m+1)}{n^*} \tag{2.6}$$

which, along with (2.2), and the facts that $\lim_{N\to\infty} a_N = Q.a.s.$, $\lim_{N\to\infty} \hat{\lambda}_N = \lambda$ *a.s.*, gives (2.3).

Again note that

$$\hat{\sigma}_n^2 = (p(n-1))^{-1} \sum_{i=1}^{n} (X_i - \bar{X}_n)' (X_i - \bar{X}_n)$$

$$\leqslant (p(n-1))^{-1} \sum_{i=1}^{n} (X_i - \underset{\sim}{\mu})'(X_i - \underset{\sim}{\mu})$$

$$= \frac{\sigma^2}{p(n-1)} \sum_{j=2}^{p(n-1)} U_j^2$$

where $\{U_j\}, j = 2, 3, \ldots$ is a sequence of independent standard normal variates. Hence from the Wiener ergodic theorem (see, Wiener [6])

$$\left\{ \sup_{n \geqslant 2} \frac{\sum_{j=2}^{p(n-1)} U_j^2}{p(n-1)} \right\}$$

has its fourth moment finite. Thus the expression on the right hand side of N/n^* in (2.6) is integrable and (2.3), together with dominated convergence theorem provides (2.4).

It follows from a result of Anscombe [1] that as $d \to 0$

$$(\bar{X}_N - \underset{\sim}{\mu})' \left(\frac{\sigma^2}{N} I_p \right)^{-1} (\bar{X}_N - \underset{\sim}{\mu}) + \frac{(\hat{\sigma}_N^2 - \sigma^2)^2}{(2\sigma^4/p(N-1))}$$

has limiting distribution χ^2 with $(p + 1)$ degrees of freedom Thus,

$$\lim_{d \to 0} P(\underset{\sim}{\theta} \in R_N) \geqslant \lim_{d \to 0} P(\underset{\sim}{\theta} \in R_N^*)$$

$$\geqslant P\left\{ \chi^2_{(p+1)} \leqslant \frac{d^2}{\lambda} (n^* - 1) \right\} = \alpha$$

in view of (2.3).

Remark. Following Robbins [3] and Starr [4], one can derive sequential procedures for simultaneous estimation of $\underset{\sim}{\mu}$ and σ^2 under the loss function

$$L_n(C) = (\bar{X}_n - \underset{\sim}{\mu})'(\bar{X}_n - \underset{\sim}{\mu}) + (\hat{\sigma}_n^2 - \sigma^2)^2 + Cn$$

where C is the known cost per unit observation. The value n_0 of n which minimizes the risk is (approximately) $C^{-1/2} \sigma (p + \sigma^2/2p)^{1/2}$. Since the stopping rule and estimation rule are highly dependent, the technique of Starr/Woodroofe [5] can be adopted to prove asymptotic risk-efficiency of the procedure.

REFERENCES

[1] Anscombe, F. J. (1952) : Large sample theory of sequential estimation, *Proc. Comb. Philos. Soc.*, **48** : 600-607.

[2] Mukhopadhyay, N. (1981) : Sequential approach to simultaneous estimation of the mean and variance, *Metrika*, **28** : 133-136.

[3] Robbins, H. (1959) : Sequential estimation of the mean of a normal population. *Prob. and Statist.* (*Harold Cramer Vol.*), John Wiley and Sons, New York.

[4] Starr, N. (1966) : On the asymptotic efficiency of a sequential procedure for estimating the mean, *Ann. Math. Statist.*, **37** : 1173-1185.

[5] Starr, N. and Woodroofe, M. (1972) : Further remarks on sequential estimation: The exponential case, *Ann. Math. Statist.*, **43** : 1147-1154.

[6] Wiener, N. (1939) : The ergodic theorem, *Duke Math. Jour.*, **5** : 1-18.